Problem 2

What is your interpretation of the initial-boundary value problem?

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 1 \end{cases} \qquad 0 < t < \infty.$$

$$\text{IC} \qquad u(x,0) = \sin(\pi x) \qquad 0 \le x \le 1$$

Can you draw rough sketches of the solution for different values of time? Will the solution come to a steady state; is this obvious?

Solution

This IBVP models the temperature in a homogeneous one-dimensional rod whose lateral side is insulated; the temperature at $x = 0$ is fixed at zero temperature, and there's a constant heat flux at $x = 1$. The initial temperature profile is given by $u = \sin(\pi x)$. To solve the IBVP, take advantage of the fact that the PDE and its associated conditions are linear. Assume that the temperature has a steady component and an unsteady component.

$$u(x,t) = v(x) + w(x,t)$$

Plug this into the PDE

$$\frac{\partial}{\partial t}[v(x) + w(x,t)] = \alpha^2 \frac{\partial^2}{\partial x^2}[v(x) + w(x,t)] \quad \rightarrow \quad w_t = \alpha^2 [v''(x) + w_{xx}]$$

and each of the conditions.

$$\begin{aligned} u(0,t) = 0 \qquad &\rightarrow \qquad v(0) + w(0,t) = 0 \\ u_x(1,t) = 1 \qquad &\rightarrow \qquad v'(1) + w_x(1,t) = 1 \\ u(x,0) = \sin(\pi x) \qquad &\rightarrow \qquad v(x) + w(x,0) = \sin(\pi x) \end{aligned}$$

To make the PDE and boundary conditions homogeneous for $w(x,t)$, let $v''(x) = 0$ and $v(0) = 0$ and $v'(1) = 1$. Then the IBVP for $w(x,t)$ is

$$\begin{aligned} &w_t = \alpha^2 w_{xx} \\ &w(0,t) = 0 \\ &w_x(1,t) = 0 \\ &w(x,0) = \sin(\pi x) - v(x). \end{aligned}$$

Start by solving the ODE for v. The general solution for $v''(x) = 0$ is

$$v(x) = C_1 x + C_2.$$

Take the derivative with respect to x.

$$v'(x) = C_1$$

Apply the boundary conditions to determine C_1 and C_2.

$$v(0) = C_2 = 0$$

$$v'(1) = C_1 = 1$$

The steady-state temperature is therefore

$$v(x) = x.$$

Using the method of separation of variables, the solution to the IBVP,

$$w_t = \alpha^2 w_{xx}$$

$$w(0,t) = 0$$

$$w_x(1,t) = 0$$

$$w(x,0) = \sin(\pi x) - x,$$

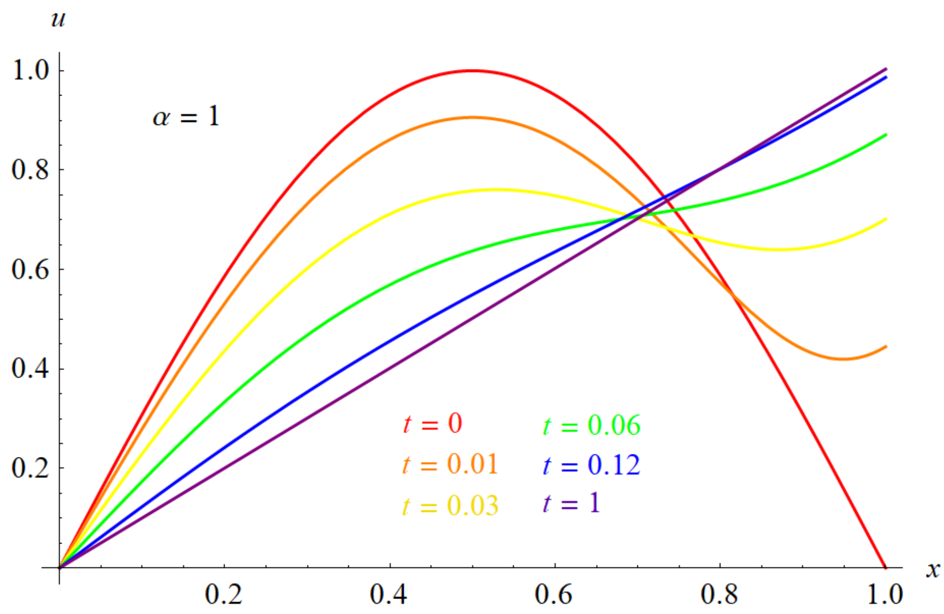
is found to be

$$w(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n[4(\pi+1)n(n-1)+\pi-3]}{(2n-3)(2n-1)^2(2n+1)} \exp\left[-\frac{\pi^2\alpha^2}{4}(2n-1)^2 t\right] \sin\left[\frac{\pi}{2}(2n-1)x\right].$$

Therefore, since $u(x,t) = v(x) + w(x,t)$,

$$u(x,t) = x + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n[4(\pi+1)n(n-1)+\pi-3]}{(2n-3)(2n-1)^2(2n+1)} \exp\left[-\frac{\pi^2\alpha^2}{4}(2n-1)^2 t\right] \sin\left[\frac{\pi}{2}(2n-1)x\right].$$

Plot this function versus x at several times for $\alpha = 1$ to illustrate the behavior of this solution.



Notice that the temperature profile is $u = \sin(\pi x)$ at $t = 0$, but over time it approaches the steady state $u = x$. The temperature is zero at $x = 0$, and the slope of the temperature curve is 1 at $x = 1$, consistent with the boundary conditions.